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Exploring the $\Delta I = 1/2$ Rule in Non-Leptonic Kaon Decays

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A satisfactory theoretical explanation for the large enhancement of the $\Delta I = 1/2$ amplitude in $K \to \pi\pi$ decays has been lacking for more than 30 years. While it has long been suspected that the charm quark, and the fact that its mass is much larger than typical hadronic scales, could be the main source of the effect, it is only now that this scenario can be tested reliably via lattice simulations of QCD. In this note we give an account of our ongoing project to investigate the mechanism behind the $\Delta I = 1/2$ rule. So far we have found indirect evidence that a significant contribution to the enhancement must indeed come from the charm quark.

1 Introduction

The decays and mixing patterns of K-mesons, mediated by the weak interaction, are among the prime sources of information on discrete symmetries in Nature. In particular, the violation of CP symmetry, which transforms particles into anti-particles, has wide-ranging implications, since it may explain why matter dominates over anti-matter in the universe. In the Standard Model, CP violation is incorporated via a complex phase in the Cabibbo-Kobayashi-Maskawa matrix, and its value can be pinned down by studying the mixing between a neutral kaon, K^0, and its anti-particle, $\bar{K}^0$. Such a process gives rise to what is called *indirect* CP violation, while CP symmetry is *directly* violated in certain non-leptonic kaon decays, such as $K \to \pi\pi$.

Despite many efforts, our theoretical understanding of non-leptonic kaon decays on a quantitative basis is still rather limited. So far, attempts to compute the amount of direct versus indirect CP violation, which is parameterised by the ratio ϵ'/ϵ have not produced credible results. A closely related problem is the failure to explain the so-called $\Delta I = 1/2$ rule: if a neutral kaon, having isospin 1/2, decays into a pair of pions, the latter can either have isospin $I = 0$ or 2. The corresponding transition amplitudes are then given by the amplitudes A_0 and A_2 (up to a phase factor). The experimentally observed decay rates yield an unexpectedly large ratio of

$$A_0/A_2 \approx 22.1, \tag{1}$$

which implies that the decay in which isospin changes by 1/2 is favoured over the $\Delta I = 3/2$ transition by a large margin, and this observation is usually called the $\Delta I = 1/2$ rule.

Quantum Chromodynamics (QCD), the gauge theory of the strong interaction, should in principle allow to explain the large enhancement of the amplitude A_0. However, theoretical calculations based on perturbative QCD in conjunction with naïve estimates of the hadronic matrix elements involved, yield a value for A_0/A_2 that turns out to be smaller by a full order of magnitude. Thus, a successful explanation of the $\Delta I = 1/2$ rule must inevitably be based on a non-perturbative treatment of QCD. Numerical simulations of QCD on a space-time lattice are designed for this task. Nevertheless, the directly computation of amplitudes for non-leptonic decays such as $K \to \pi\pi$ remains a great challenge.

A cornerstone of our computational strategy is the matching of QCD to an effective low-energy description, called Chiral Perturbation Theory (ChPT). Since the amplitudes A_0, A_2 can be expressed in terms of the leading-order low-energy constants (LECs), g_1^+ and g_1^-, of the interaction Hamiltonian which describes weak non-leptonic kaon decays in ChPT, the main idea is to compute the LECs in lattice simulations. This can be achieved by comparing the predictions of ChPT for suitably chosen two- and three-point correlation functions in a finite volume with the analogous ones computed in lattice QCD for small masses and momenta.

In this note we present a progress report of our project [1–6] to investigate the mechanism of the $\Delta I = 1/2$ rule. In particular we seek to clarify whether the observed large enhancement in A_0 over A_2 has a single origin or is the result of an accumulation of several moderately large effects. To this end we specifically concentrate on the rôle of the charm quark. The fact that the mass of the latter of around 1.3 GeV is much greater than typical low-energy scales of a few hundred MeV, has led to the conjecture that its decoupling may produce a large enhancement of A_0, due to the appearance of additional operators with large hadronic matrix elements.[7] In our strategy we keep the charm quark as an "active" degree of freedom, which – in contrast to most other studies – is not integrated out from the theory.

In the first part of the project the LECs g_1^+, g_1^-, and, in turn, the amplitudes A_0 and A_2 are determined for the unphysical situation where the charm is degenerate with the light quark, $m_c = m_u = m_d = m_s$. In a second step we envisage monitoring the amplitudes for heavier charm, i.e. $m_c > m_u = m_d = m_s$.

2 Computational Strategy

The main ingredients of our strategy are described in [1,3] and are briefly summarised here. In the first stage of the calculation we focus on the case of a light and degenerate charm quark. Decays such as $K \to \pi\pi$ in which strangeness changes by one unit are described in QCD via the interaction Hamiltonian

$$\mathcal{H}_\mathrm{w} = \sqrt{2} G_\mathrm{F} (V_{us})^* V_{ud} \left\{ k_1^+ \mathcal{Q}_1^+ + k_1^- \mathcal{Q}_1^- \right\}, \tag{2}$$

where G_F is the Fermi constant and V_{us}, V_{ud} denote elements of the quark mixing matrix. The operators $\mathcal{Q}_1^\pm$ are expressed in terms of quark fields according to

$$\mathcal{Q}_1^\pm = \Big\{ (\overline{s}\gamma_\mu P_- u)(\overline{u}\gamma_\mu P_- d) \pm (\overline{s}\gamma_\mu P_- d)(\overline{u}\gamma_\mu P_- u) \Big\} - (u \to c). \tag{3}$$

The Wilson coefficients $k_1^\pm$ in the above expression absorb short-distance effects and can be computed reliably in perturbation theory.

In the chiral effective theory (ChPT), the corresponding Hamiltonian reads

$$\mathcal{H}_{\rm w}^{\rm ChPT} = 2\sqrt{2}G_{\rm F}(V_{us})^{*}V_{ud}\left\{g_1^{+}[\widehat{\mathcal{O}}_1^{+}] + g_1^{-}[\widehat{\mathcal{O}}_1^{-}]\right\}, \tag{4}$$

where the operators $\widehat{\mathcal{O}}_1^{+}$ and $\widehat{\mathcal{O}}_1^{-}$ are now expressed in terms of Goldstone boson fields rather than quark degrees of freedom. The *a priori* unknown LECs $g_1^{\pm}$ incorporate the short-distance effects from the strong interaction in this effective low-energy description. In the chiral limit, they are directly related to the amplitudes A_0 and A_2 via

$$\frac{A_0}{A_2} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{3}{2}\frac{g_1^{-}}{g_1^{+}}\right). \tag{5}$$

Thus, the determination of the LECs in lattice simulations of QCD gives a handle to compute the ratio of amplitudes. The key quantities which allow to determine $g_1^{\pm}$ are correlation functions of the four-quark operators $\mathcal{Q}_1^{\pm}$ and the left-handed current, i.e.

$$C_1^{\pm}(x_0,y_0) = \sum_{\vec{x},\vec{y}}\left\langle(\overline{d}\gamma_0 P_{-}u)(x)\,\mathcal{Q}_1^{\pm}(0)\,(\overline{u}\gamma_0 P_{-}s)(y)\right\rangle, \quad P_{-} = \tfrac{1}{2}(1-\gamma_5). \tag{6}$$

We are particularly interested in ratios of correlation functions, such as

$$R^{\pm}(x_0,y_0) = \frac{C_1^{\pm}(x_0,y_0)}{C(x_0)C(y_0)}, \tag{7}$$

where the two-point function of the left-handed current is given by

$$C(x_0) = \sum_{\vec{x}}\left\langle(\overline{d}\gamma_0 P_{-}u)(x)\,(\overline{u}\gamma_0 P_{-}s)(0)\right\rangle. \tag{8}$$

With these definitions one may formulate a matching condition, which allows to express the unknown LECs in terms of the ratios $R^{\pm}$, the latter of which are computable in lattice simulations. At large Euclidean times x_0, y_0 the ratios $R^{\pm}$ are proportional to $g_1^{\pm}$:

$$k_1^{\pm}(Z^{\pm}/Z_{\rm A}^2)\,R^{\pm} = \left\{1 + K^{\pm}\right\}\,g_1^{\pm}, \tag{9}$$

where the $K^{\pm}$ parameterise chiral corrections. The latter manifest themselves in terms of a dependence of the ratios $R^{\pm}$ on the volume (i.e. the box size L) and the quark mass. Thus, by computing $R^{\pm}$ for a range of finite volumes and/or quark masses, knowledge of $K^{\pm}$ serves to extract the LECs via eq. (9). Two kinematical regimes of QCD are of particular importance in this regard: the so-called ϵ-regime [8] is characterised by considering the chiral limit in a finite volume, so that the pion's correlation length exceeds the box size L, and the latter is the only scale left in the theory. Moreover, correlation functions may depend strongly on the topological properties of the gauge field in this regime [9]. On the other hand, in the so-called p-regime the pions still fit in the box, and chiral corrections can be computed as a power series in $(m_{\pi}L)^2$.

In our strategy we combine numerical data for $R^{\pm}$ from the two kinematical regimes in order to have better control over the mass and volume dependence, which, in turn, should lead to a more reliable determination of the LECs. The expressions for the chiral corrections $K^{\pm}$ at next-to-leading order in the chiral expansion have been published in[1,10] and[4] for the ϵ- and p-regimes, respectively.

In addition to the chiral corrections, the matching condition, eq. (9), also contains short-distance corrections. These are given by the Wilson coefficients $k_1^{\pm}$, which can be computed reliably in perturbation theory, as well as the renormalisation factors $Z^{\pm}$ of the four-quark operators and Z_{A} of the axial current. The latter must be considered in the matching condition to account for the proper renormalisation of the operators whose correlation functions appear in the ratios $R^{\pm}$. As part of our project we have determined the renormalisation factors non-perturbatively [5], by applying the technique originally proposed in [11].

3 Numerical Simulations and Results

The most important ingredient in our simulations is the use of a fermionic discretisation which preserves chiral symmetry at finite lattice spacing. As pointed out in [12], this is the case for any Dirac operator which satisfies the Ginsparg-Wilson relation.[13]

Preserving chiral symmetry at all stages of the calculation is indispensable to allow for a reliable matching to ChPT. However, it comes at a price, since the implementation of the Neuberger-Dirac operator is numerically very costly. This is further exacerbated in the chiral regime, which is susceptible to numerical instabilities as well as strong statistical fluctuations, which are generated by the appearance of arbitrarily small eigenvalues of the Dirac operator.

In our simulations we employ the Neuberger-Dirac operator.[14] We have developed several numerical techniques in order to control the effects of the small eigenmodes. They include the determination of the topological index ν via the counting of zero modes, in the course of which the latter can also be determined with sufficient numerical accuracy.[15] Second, one can speed up the numerical inversion of the Neuberger-Dirac operator considerably by applying "low-mode preconditioning".[15] Finally, the large statistical fluctuations encountered for small quark masses and, in particular, in the ϵ-regime can be tamed by applying "low-mode averaging".[16,17] Without going into detail here, we refer the reader to the original articles cited above.

Our simulations have been performed in the quenched approximation, such that the effects of dynamical quarks are unaccounted for. Despite the fact that one incurs an unknown systematic error, one may argue that the use of the quenched approximation is quite sufficient for the sake of investigating the origins of the $\Delta I = 1/2$ rule, since one tries to explain an enhancement of a decay amplitude by a full order of magnitude, while typical quenching effects amount to $10 - 15\%$. Our main results have been obtained on lattices of size $16^3{\cdot}32$ for a value of the bare coupling $\beta \equiv 6/g_0^2 = 5.8485$, which corresponds to a lattice spacing in physical units of $a \simeq 0.12\,\mathrm{fm}$. In the ϵ-regime we computed the ratios $R^{\pm}$ on 746 configurations for two values of the quark mass. Results were divided into bins of fixed topological charge $|\nu|$, and a weighted average over the results computed in each bin was taken inside an interval $2 \leq |\nu| \leq 10$. In the p-regime $R^{\pm}$ was determined on 197 configurations at four values of the quark mass without distinguishing configurations in different topological sectors. Plots of the (unrenormalised) ratios $R^{\pm}$ obtained in both kinematical regimes as a functions of the bare quark mass are shown in Fig. 1.

The fits shown in the figure yield estimates for the bare LECs. After including the Wilson coefficients and the renormalisation factors, we obtain the final results [3]

$$g_1^{+} = 0.51(3)(5)(6), \qquad g_1^{-} = 2.6(1)(3)(3), \tag{10}$$

which apply in the case of a light degenerate charm quark considered in our study. The quoted errors arise from statistics, the matching to ChPT and from the uncertainty in the non-perturbative determination of the renormalisation factors. From these results one may

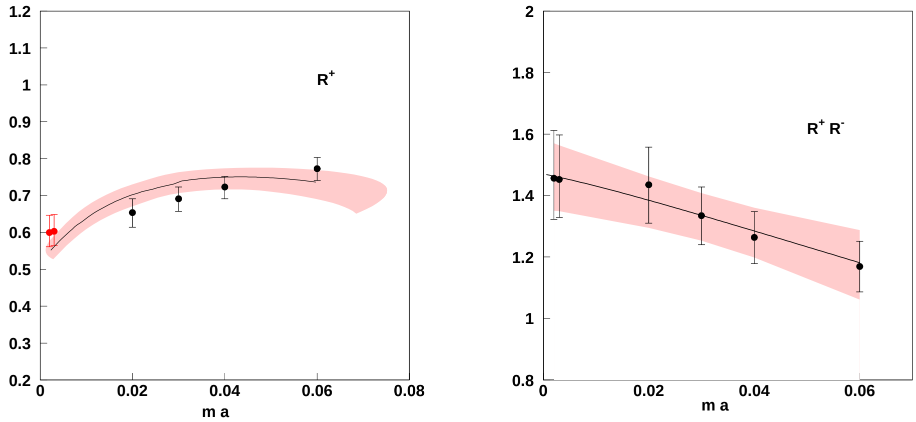

Figure 1. The ratio R^+ and the product R^+R^-. The two left-most points in each plot represent the data obtained in the ϵ-regime, while the remaining points lie in the p-regime. The shaded red bands denote the error from a joint fit of the data in both kinematical regimes to the expressions of ChPT.

compute the amplitudes A_0 and A_2 and compare to the physical result. First, one observes that $g_1^- \gg g_1^+$ so that our results indeed imply a clear hierarchy, producing an enhancement of A_0 over A_2 due to non-perturbative effects. Moreover, our findings indicate that the physical $\Delta I = 3/2$ amplitude A_2 is quite accurately reproduced by our result for g_1^+. However, the observed hierarchy is not sufficient to fully explain the physical enhancement of A_0, which is underestimated by a factor 4. Hence, our results support the conjecture that the decoupling of the charm quark plays a significant rôle for the $\Delta I = 1/2$ rule, despite the fact that our findings so far represent only indirect evidence for this scenario.

4 $g_1^\pm$ from Zero-Mode Wavefunctions

In the presence of gauge fields with non-trivial topology, $\nu \neq 0$, the (massless) Dirac operator exhibits $|\nu|$ zero modes of definite chirality, according to the Atiyah-Singer index theorem. This can be exploited for an alternative determination of $g_1^\pm$, by considering three-point functions of $\mathcal{Q}_1^\pm$ with the left-handed currents replaced by the corresponding pseudoscalar densities, i.e.

$$(\overline{u}\gamma_0 P_- s)(x) \to i(\overline{u}\gamma_5 s)(x), \qquad (\overline{d}\gamma_0 P_- u)(x) \to i(\overline{d}\gamma_5 u)(x). \tag{11}$$

In the ϵ-regime and for $\nu \neq 0$, the corresponding correlation functions may develop poles in $1/(mV)$, whenever the zero modes give a non-vanishing contribution. Since the residues of the poles admit a chiral expansion parameterised in terms of the same set of LECs, the alternative strategy is to compute the residues rather than the correlation functions themselves in the simulation.

The advantage of this strategy is that the zero modes can be obtained with relatively little numerical effort in the course of our calculation. Moreover, the systematic effects in this approach will be different from those of our standard method, thereby providing an independent cross-check on our findings described above.

To be more specific, we list a few more definitions. We consider the residue of the three-point function

$$\mathcal{C}^{\sigma}_{1;\,\nu} = \lim_{m\to 0}(mV)^2 \sum_{\vec{x},\vec{y}} \left\langle (\overline{d}\gamma_5 u)(x)\, \mathcal{Q}^{\sigma}_1(z)\, (\overline{u}\gamma_5 s)(y)\right\rangle_\nu$$
$$\equiv A_\nu(x_0 - z_0, y_0 - z_0) + \sigma \tilde{A}_\nu(x_0 - z_0, y_0 - z_0), \quad \sigma = \pm, \tag{12}$$

where A_ν and $\tilde{A}_\nu$ can be expressed in terms of zero-mode wavefunctions. For instance, if the zero modes have negative chirality, the expression for A_ν reads

$$A_\nu = \frac{1}{L^3} \sum_{\vec{x},\vec{y},\vec{z}} \left\langle \sum_{i\in\mathcal{K}} v_i^\dagger(x) S(x,z)\gamma_\mu P_- v_i(z) \sum_{j\in\mathcal{K}} v_j^\dagger(y) S(y,z)\gamma_\mu P_- v_j(z) \right\rangle_\nu, \tag{13}$$

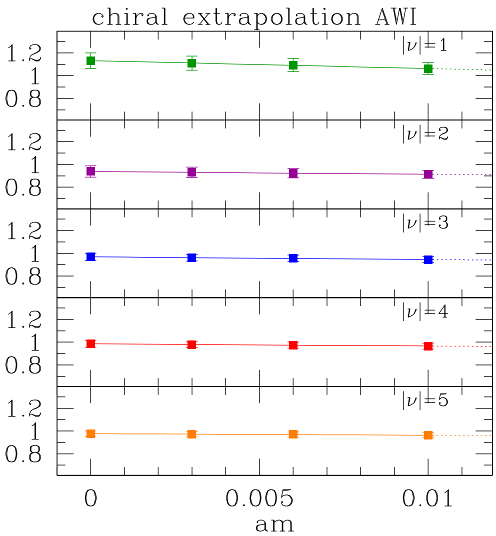

Figure 2. Verification of the chiral Ward identity, based on correlation functions saturated by zero modes, in various topological sectors.

where v_i is a zero mode. A similar expression can be derived for $\tilde{A}_\nu$. Equation (13) illustrates that a considerable amount of CPU time can be saved if one computes the above residues of correlation functions, in which some of the quark propagators are replaced by the zero mode contributions. In order to determine A_ν and $\tilde{A}_\nu$, the number of inversions of the Dirac operator is equal to twice the topological charge, i.e. $2|\nu|$, (with sources $v_i(x)$ and $v_i(y)$, since x_0 and y_0 need to be fixed). At the same time one can average over all spatial positions of the three sources, thereby reducing statistical fluctuations. Such averaging was only possible in the standard method of[1] through low-mode averaging (LMA), and only for the contribution of the low-modes. The price of LMA is $12 + 2 \times N_{\rm low}$ inversions, where $N_{\rm low}$ is the number of low modes treated separately. Typically $N_{\rm low}$ can be as large as 20, and hence the numerical cost of LMA is usually far greater than that required for the computation of the residues, where $|\nu|$ normally is not larger than 5.

In order to formulate the matching condition, we also consider the residues of the two-point functions of the pseudoscalar density and the left-handed axial current:

$$\mathcal{C}_\nu(x_0 - z_0) \equiv \lim_{m\to 0}(mV) \sum_{\vec{x}} \left\langle (\overline{d}\gamma_5 u)(x)(\overline{u}\gamma_0 P_- d)(z)\right\rangle_\nu . \tag{14}$$

In practice it is advantageous to consider temporal derivatives of the correlators, and hence we define the ratio of correlation functions as

$$\mathcal{R}^{\pm}_\nu(x_0, y_0) = \frac{\partial_{x_0}\partial_{y_0}\mathcal{C}^{\pm}_{1;\,\nu}(x_0 - z_0, y_0 - z_0)}{\partial_{x_0}\mathcal{C}_\nu(x_0 - z_0)\partial_{y_0}\mathcal{C}_\nu(y_0 - z_0)}, \tag{15}$$

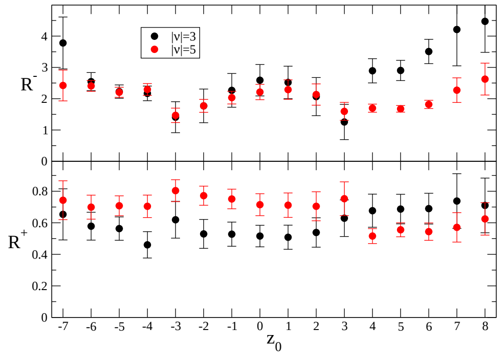

Figure 3. The ratios $\mathcal{R}_\nu^\pm$ for $|\nu| = 3$ and 5 computed on lattice of size 16^4 at $\beta = 5.8458$ for $x_0 = 5$ and $y_0 = 11$. The accumulated statistics amounts to only 27 and 24 configurations for $|\nu| = 3$ and 5, respectively.

while the matching condition now reads

$$k_1^\pm (Z^\pm/Z_\mathrm{A}^2)\, \mathcal{R}_\nu^\pm = \left\{1 + \mathcal{K}_\nu^\pm\right\}\, g_1^\pm. \tag{16}$$

The chiral corrections $\mathcal{K}_\nu^\pm$ have been worked out in ChPT at next-to-leading order.[6] It remains to compute the ratios $\mathcal{R}_\nu^\pm$ in simulations and extract the LECs $g_1^\pm$ through eq. (16).

An important consistency check of the procedure, which involves the calculation of two-point functions only, is the verification of the chiral Ward identity. In the chiral limit one expects

$$Z_\mathrm{A} \partial_{x_0} \mathcal{C}_\nu(x_0 - y_0) = \lim_{m\to 0} (m^2 V) \sum_{\vec{x}} \left\langle (\overline{d}\gamma_5 u)(x)(\overline{u}\gamma_5 d)(y) \right\rangle_\nu . \tag{17}$$

In practice, the above correlators are evaluated for small, but non-zero quark masses. After taking the renormalisation factor of the axial current, Z_A, into account [18], the ratio of the two sides of eq. (17) is expected to extrapolate to unity, and indeed this is what we observe (see Fig. 2). The calculation of the residues of three-point functions on lattices of size 16^4 and 24^4 is currently under way. A plot of the preliminary data for the ratios $\mathcal{R}_\nu^\pm$ is shown in Fig. 3.

5 Conclusions and Outlook

Ginsparg-Wilson fermions offer an attractive framework to tackle one of the most notorious and long-standing problems in the physics of hadrons. We have completed the first part of our project to investigate the mechanism of the $\Delta I = 1/2$ rule and found evidence for a significant non-perturbative enhancement of the amplitude A_0 in the artificial situation of a light and degenerate charm quark. This represents indirect evidence for the scenario that a substantial part of the enhancement should be due to the decoupling of the charm quark. An important cross-check of these findings, based on correlators saturated with zero modes is under way. As the latter method is numerically cheaper, it will be advantageous to study the case of a heavy charm quark also in this framework. The additional correlation functions have already been programmed, and also the required chiral corrections are being worked out[19].

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